4.5 The dimension of a Vector Space.

Lemma- Any set in $\mathbb{R}^n$

$$S = \{\vec{v}_1, \ldots, \vec{v}_p\} \quad \underline{p > n}.$$

is linearly dependent.

Proof.-

Want to prove, there exists $c_1, \ldots, c_p$ not all zeros such that

$$c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p = \vec{0}.$$

First, we consider the linear combination

$$c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p = \vec{0}$$

If we express this as a system of equations in matrix form

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n & \cdots & \vec{v}_p \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots_n \\ 0 \end{bmatrix}$$

or

$$\begin{array}{ccccc} \vec{V}_1 & \vec{V}_2 & & \vec{V}_n & \vec{V}_p \\ \end{array}$$

$$\begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} & \cdots & V_{1p} \\ V_{21} & V_{22} & & V_{2n} & & V_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ V_{n1} & V_{n2} & & V_{nn} & & V_{np} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

more Columns than rows $\Rightarrow$ there exists at least a free variable $\Rightarrow$ infinitely many solutions $\Rightarrow$ there are nonzero c's. $\Rightarrow$ $S = \{\vec{V}_1, \ldots, \vec{V}_p\}$ lin dependent.

Consider the set of polynomials

$$S = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \{1+2t, 5+3t, -1+4t\}$$

show that they are linearly dependent.

A way to do this is to consider the transformation $(B = \{1, t\})$.

$$T: \mathbb{P}_1 \longrightarrow \mathbb{R}^2$$

$$\vec{p} = a_0 + a_1 t \rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = [\vec{p}]_B$$

Isomorphism btw. $\mathbb{P}_1$ and $\mathbb{R}^2$.

And then study the linear dependence relationship between the corresponding vectors in $\mathbb{R}^2$ images of

In fact,

$$T(\vec{p}_1) = T(1+2t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [\vec{p}_1]_B$$

$$T(\vec{p}_2) = T(5+3t) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [\vec{p}_2]_B$$

$$T(\vec{p}_3) = T(-1+4t) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} = [\vec{p}_3]_B$$

Now, the set $\tilde{S} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$

is a set of 3 vectors in $\mathbb{R}^2$.

Then, according to the previous lemma

$\check{S}$ is a lin. dependent set in $\mathbb{R}^2$.

Therefore, the corresponding set of vectors in $\mathbb{P}_1$

$S = \{ \vec{p}_1, \vec{p}_2, \vec{p}_3 \}$ is also lin. dependent, because

T is an isomorphism between $\mathbb{P}_1$ and $\mathbb{R}^2$.

4.5 The Dimension of a Vector Space.

Thm 9.- V vector space

$B = \{\vec{b}_1, .., \vec{b}_n\}$ is a basis for V.

Any subset of V containing more than n vectors must be linearly dependent.

Proof.- Consider a subset $S = \{\vec{u}_1, .., \vec{u}_p\} \subset V$, $p > n$

Also consider the corresponding set $S' \subset \mathbb{R}^n$ formed by the coordinate vectors

$$S' = \{[\vec{u}_1]_B, .., [\vec{u}_p]_B\} \quad p > n$$

According to our previous lemma, this set is linearly dependent. Therefore, there exists coefficients $c_1 .., c_p$ not all zero such that

$$c_1 [\vec{u}_1]_B + \dots + c_p [\vec{u}_p]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{\mathbb{R}^n}.$$

Using the linearity of the transformation (Coord. mapping).

$$\left[c_1\vec{u}_1 + \cdots + c_p\vec{u}_p\right]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{\mathbb{R}^n}$$

But,
$$\vec{0} \in V \xrightarrow{[\;]_B} \vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{\mathbb{R}^n}$$

It means
$$[\vec{0}]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{\mathbb{R}^n}$$

Now, Coord. mapping is one-to-one then $[\vec{u}]_B = [\vec{v}]_B \Rightarrow \vec{u} = \vec{v}$.

$$c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$$ and not all c's are zero.

Thus, $S = \{\vec{u}_1, .., \vec{u}_p\}$ $p > n$ is lin. dependent.

<u>Thm 10.</u> If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

<u>Proof.</u>- Consider two basis $B = \{\vec{b}_1, .., \vec{b}_n\}$ and $D = \{\vec{d}_1, .., \vec{d}_p\}$ of V. <u>We want to prove</u> $\boxed{p = n.}$

<u>We know</u> both B and D are 1) Lin. indep. 2) Span V

Since B is a basis of n vectors any set with more than n vectors must be linearly dep. (thm 9)
Therefore, for D to be lin. indep.

$$p \leq n.$$

Similarly, if D is a basis of p vectors and B is a set of n vectors lin. indep.

$$n \leq p.$$

$$\Rightarrow \quad \boxed{n = p} \quad \checkmark$$

<u>Definition</u>

i) If the vector space is spanned by a finite set then V <u>is said to be finite-dimensional.</u>

ii) <u>The dimension</u> of a finite-dimensional <u>vector space</u> is the number of vectors in a basis of V.

iii) The dimension of $V = \{\vec{0}\}$ is zero.

iv) If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Thm 11 $H \subset V$, V finite-dimensional Vector space.
H Subspace of V.

i) Any lin. indep. set in H can be expanded to a basis

ii) $$\dim(H) \leq \dim(V).$$

Proof.- Based on thm 9. and thm 4.

Thm 12 V p-dimensional Vector space

i) Any lin. indep. set of exactly p vectors is automatically a basis for V.

ii) Any set S of exactly p elements such that
$$\text{Span}(S) = V$$
is automatically a basis for V.

Proposition The dimension of Nul A is the number of free Variables in the equation $A\vec{x} = \vec{0}$, and the dimension of Col A is the number of pivot Columns in A.

Example 3 (Sect. 4.2 boon)

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 \quad - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5, \end{cases} \quad x_2, x_4, x_5 \text{ free Variables.}$$

$$\Rightarrow \quad \tilde{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{lin. indep. set}}$$

also spans Nul A.